Lecture - 14

In this lecture we'll study about Normal subgroups and Quotient groups.
Recall from Lec. 9 that if G is a group and $H < G$ then aH might not be equal to Ha.
It turns out that when $aH = Ha$ for all $a \in G$, then those subgroups H are extremely important. This was observed by Galois around 200 years ago when he was just 18!

Definition [Normal Subgroup]

A subgroup $H \leq G$ is called a normal subgroup if $aH = Ha$ $\forall$ $a \in G$. We denote this by $H \triangleleft G$.

Remark:- Note that $aH = Ha$ means that if we look at ah, for $h \in H$ then $\exists$ an element $h' \in H$ such that $ah = h'a$. h' might be same as h but we cannot gurantee that.

Just like the subgroup test, we have a normal subgroup test for checking if a subgroup is normal or not. In fact, there are many equivalent formulations of the definition of a normal subgroup and we'll see some of them.

<u>Normal Subgroup test</u>

<u>Theorem 1</u> Let $H \leq G$. Then $H \triangleleft G$ if and only if $aHa^{-1} \subseteq H \quad \forall \ a \in G$.

<u>Remark</u> :- Note that $aHa^{-1} = \{ aha^{-1} \mid h \in H \}$

Proof $\Rightarrow$ Suppose $H \triangleleft G$. This means that

$aH = Ha \quad \forall a \in G$.

Let $aha^{-1} \in aHa^{-1}$. Since $ah = h'a$ for some h'

$\Rightarrow aha^{-1} = (h'a)a^{-1} = h' \in H$.

$\Leftarrow$ Suppose $aHa^{-1} \subseteq H \ \forall a \in G$. We want to prove that $aH = Ha \ \forall a \in G$.

Since $aHa^{-1} \subseteq H \Rightarrow (aHa^{-1}).a \subseteq Ha$

$\Rightarrow aH \subseteq Ha$.

On the other hand $a^{-1}H(a^{-1})^{-1} \subseteq H$

$\Rightarrow a^{-1}Ha \subseteq H. \Rightarrow a(a^{-1}Ha) \subseteq aH$

$\Rightarrow Ha \subseteq aH$

So, $Ha = aH$.

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Examples

① Let G be an abelian group. Then every subgroup of G is a normal subgroup.

② Let G be a group and consider the center of the group $Z(G)$. Then $Z(G) \trianglelefteq G$. [Prove this]

③ Consider S_n. Then the alternating group A_n, consisting of even permutations in S_n is a normal subgroup. It's pretty easy to see if we use the normal subgroup test and the fact that the inverse of an odd (or even) permutation remains an odd (respectively even) permutation.

④ Consider D_n, the dihedral group of order $2n$. The subgroup of D_n consisting only of rotations is a normal subgroup of D_n.

Again recall from Lec. 9 that Ha or aH might not be a subgroup of G, even though H is.

So if $H \leq G$ and $K \leq G$ then $HK = \{hk \mid h \in H, k \in K\}$ might not be a subgroup of G, even though both H and K are. The next theorem gurantees when HK is a subgroup :-

<u>Theorem</u> 2 Let $H \trianglelefteq G$ and $K \leq G$. Then $HK \leq G$.

<u>Proof</u> On Assignment 3.

Another important property of normal subgroups is that if a subgroup is normal then the set of it's cosets (left or right) is itself a group called the quotient group.

To understand this, let's see some examples.

__Example 1__ Consider $\mathbb{Z}$ and it's subgroup $3\mathbb{Z}$. What are the left cosets of $3\mathbb{Z}$?

They are

$0+3\mathbb{Z} = \{0, \pm 3, \pm 6, \dots\}$

$1+3\mathbb{Z} = \{\dots, -5, -2, 1, 4, \dots\}$

$2+3\mathbb{Z} = \{\dots, -4, -1, 2, 5, \dots\}$

These are the only left cosets of $3\mathbb{Z}$ because if $k \in \mathbb{Z}$ and we look at $k+3\mathbb{Z}$, then we first write $k = 3q + r$, $r = 0, 1$ or 2.

Then $k+\mathbb{Z} = 3q+r+3\mathbb{Z} = r+3\mathbb{Z}$ as

$3q \in 3\mathbb{Z}$.

So the set of left cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ is

$\mathcal{J} = \{0+3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\}$.

We can make $\mathcal{F}$ into a group by defining

$$(a+3\mathbb{Z})+(b+3\mathbb{Z}) = (a+b)+3\mathbb{Z}.$$

One can check that $\mathcal{F}$ is a group with this operation. Note that $3\mathbb{Z} \trianglelefteq \mathbb{Z}$.

This is happening because of the following theorem which we'll prove in the next lecture.

<u>Theorem 3</u> (Quotient Group)

Let G be a group and H be a normal subgroup of G. The set

$$\frac{G}{H} = \{aH \mid a \in G\}$$ is a group under the operation $(aH).(bH) = abH$.

$\frac{G}{H}$ is called a quotient or factor group.

Exercise Consider D_4 and consider the center of $D_4 = \{R_0, R_{180}\}$. Find all the left cosets of $Z(D_4)$ in D_4. Show that this is a group with the operation given in the above theorem. Note that $Z(D_4) \triangleleft D_4$.

Take any other subgroup of D_4 which is not normal and show that the theorem 3 fails.